Calc 3 1.9.21

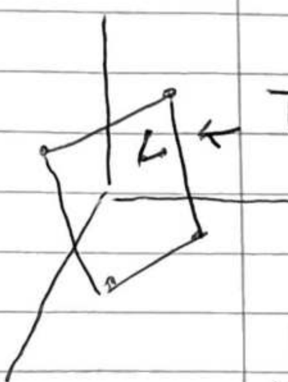
Dot Product

$\vec{v}\cdot\vec{u}=0$ when orthogenal

12.4: Cross product

* Everything today is in $\mathbb{R}^3$

Goal - Given two vectors, construct a third (ideally orthogonal to both)



To find a vector perpendicular to the plane, we find one perpendicular to the two vectors on the plane.

How?! Let $\vec{u}=(u_1, u_2, u_3)$, $\vec{v}=(v_1, v_2, v_3)$

and desired vector $\vec{w}=(w_1, w_2, w_3)$.

$\vec{w}$ must be orthogonal to both $\vec{u}$ and $\vec{v}$.

① $0=\vec{w}\cdot\vec{u} = w_1u_1+w_2u_2+w_3u_3$

② $0=\vec{w}\cdot\vec{v} = w_1v_1+w_2v_2+w_3v_3$

Multiply eq. 1 by v_3 and eq. 2 by u_3 to get two new equations

①* $0 = w_1(u_1v_3)+w_2(u_2v_3)+w_3(u_3v_3)$

②* $0 = w_1(v_1u_3)+w_2(u_3v_2)+w_3(u_3v_3)$

Subtracting 2* from 1*

$0=v_3(\vec{w}\cdot\vec{u})-u_3(\vec{w}\cdot\vec{v})$

$0=w_1(u_1v_3-u_3v_1)+w_2(u_2v_3-u_3v_2)$

$=-w_1(-u_1v_3+u_3v_1)+w_2(u_2v_3-u_3v_2)$

Solution: $w_1=u_2v_3-u_3v_2$

$w_2=-(u_1v_3-u_3v_1)$

Aside: easiest solution to $-ax+by=0$ is $x=b$, $y=a$

Plugging in to initial problem yields $w_3=u_1v_2-u_2v_1$

The solution assumes $u_3 \neq 0$

$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$
$$= \left(u_2v_3 - u_3v_2,\ -(u_1v_3 - u_3v_1),\ u_1v_2 - u_2v_1\right)$$

Exercise → To verify, check if dot products = 0

→ Linear Algebra approach - determinants

Def. The determinant of a 2x2 matrix is $ad-bc$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

& Det of a 3x3 matrix is

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$a\left(\det\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\left(\det\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\left(\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$$

ex. $$\det\begin{bmatrix} 1 & -2 & 3 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad 1\left(\det\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}\right) = (-1+2) = 1$$

The vector is a symbolic determinant

$$\begin{array}{r} \\ \vec{u} \rightarrow \\ \vec{v} \rightarrow \end{array} \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \rightarrow i(u_2v_3 - v_2u_3) - j(u_1v_3 - v_1u_3) + k(u_1v_2 - u_3v_1)$$

$$\langle (u_2v_3 - u_3v_2), -(u_1v_3 - v_1u_3), (u_1v_2 - u_3v_1) \rangle$$

Cross product of $\vec{u}$ with $\vec{v}$ is

and ~~there~~ is orthogonal to both vectors.

Cross product uses two vectors in $\mathbb{R}^3$ to produce a vector in $\mathbb{R}^3$

Properties of the cross product

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, and $c \in \mathbb{R}$

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $(\vec{u} + \vec{b}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{w}\,\vec{b}$
5. $u \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
6. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\,\vec{w}$

Geometric properties

1. $\vec{u} \times \vec{v}$ is orthogonal to both vectors
2. $|\vec{u} \times \vec{v}|$ is $|\vec{u}| \cdot |\vec{v}| \sin(\theta)$
3. parallel if cross product $= \vec{0}$